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RELATION BETWEEN ALGEBRAIC STRUCTURES

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ABSTRACT

In this paper, the authors developed the relations between different classes of algebraic structures such as BP/BM/BN/BZ/QS/B/Q – algebras with the family of BF – algebras i.e., BF/BF_1/BF_2 – algebras in detail. Several theorems were proved by imposing different types of necessary conditions on these algebras and provided examples wherever necessary. Authors also established some results by introducing e – commutative law on the family of BF – algebras

Keywords: *BF– Algebra, BF_1 – algebra, BF_2 – algebra, BM – algebra, BN – algebra and e – Commutative law.*

I. INTRODUCTION

Andrez Walendziak [13] introduced the notion of BF – algebra, which is a generalization of B – algebra and also extended BF – Algebra to and thus studied two new algebras i.e., BF_1 – algebra and BF_2 – algebra. C.B.Kim and H.S.Kim [9] introduced the notion called BG – algebra which is a generalization of B – algebra. Y. B. Jun et al. [3] introduced the notion called BH – algebra, which is a generalization of BCI/BCK/BCH –algebras. Motivated by these algebraic structures, authors have come up with some interesting findings and believe that these results may be a contribution to the earlier theories of proportional calculi, based on which Imai and Iseki had introduced the two classes of algebras BCK and BCI [4, 5, 6, 7]. Throughout this paper authors mainly concentrated on establishing relations between families of e – commutative BF – Algebras with other classical algebras.

II. PRELIMINARIES

Definition 2.1. [13] Let X be a non-empty set equipped with a binary operation $*$ and fixed element e. Then the algebraic structure $(X,*,e)$ is said to be a BF – algebra, if it satisfies the following axioms, for all $x, y \in X$
(I) $x * x = e$
(II) $x * e = x$
(BF) $e * (x * y) = y * x$.

Definition 2.2. [9] Let X be a non-empty set equipped with a binary operation $*$ and fixed element e. Then the algebraic structure $(X,*,e)$ is said to be a BG – algebra, if it satisfies the following axioms, for all $x, y \in X$
(I) $x * x = e$
(II) $x * e = x$
(BG) $(x * y) * (e * y) = x$.

Definition 2.3.[3] Let X be a non-empty set equipped with a binary operation $*$ and fixed element e. Then the algebraic structure $(X,*,e)$ is said to be a BH – algebra, if it satisfies the following axioms, for all $x, y \in X$
(I) $x * x = e$
(II) $x * e = x$
(BH) $x * y = e$ and $y * x = e$ imply that $x = y$.

Definition 2.4. [13] Let X be a non-empty set equipped with a binary operation $*$ and fixed element e. Then the algebraic structure $(X,*,e)$ is said to be a BF_1 – algebra, if it satisfies the following axioms, for all $x, y \in X$

(I) $x * x = e$
(II) $x * e = x$
(BF) $e * (x * y) = y * x$
(BG) $(x * y) * (e * y) = x$

Definition 2.5. [13] Let X be a non-empty set equipped with a binary operation $*$ and fixed element e. Then the algebraic structure $(X, *, e)$ is said to be a BF_2 – algebra, if it satisfies the following axioms, for all $x, y \in X$
(I) $x * x = e$
(II) $x * e = x$
(BF) $e * (x * y) = y * x$
(BH) $x * y = e$ and $y * x = e$ imply that $x = y$

Definition 2.6. [2] Let X be a non-empty set equipped with a binary operation $*$ and fixed element e. Then the algebraic structure $(X, *, e)$ is said to be a BP – algebra, if it satisfies the following axioms, for all $x, y, z \in X$.
(I) $x * x = e$
(BP_1) $(x * (x * y)) = y$
(BP_2) $(x * z) * (y * z) = x * y$.

Definition 2.7. [8] Let X be a non-empty set equipped with a binary operation $*$ and fixed element e. Then the algebraic structure $(X, *, e)$ is said to be a BM – algebra, if it satisfies the following axioms, for all $x, y, z \in X$
(I) $x * x = e$
(II) $x * e = x$
(BM) $(z * x) * (z * y) = y * x$.

Definition 2.8. [10] Let X be a non-empty set equipped with a binary operation $*$ and fixed element e. Then the algebraic structure$(X, *, e)$ is said to be a BN – algebra, if it satisfies the following axioms, for all $x, y, z \in X$
(I) $x * x = e$
(II) $x * e = x$
(BN) $(x * y) * z = (e * z) * (y * x)$

Definition 2.9. [14] Let X is a non-empty set equipped with a binary operation $*$ and fixed element e. Then the algebraic structure $(X, *, e)$ is said to be a BZ – algebra, if it satisfies the following axioms, for all $x, y, z \in X$
(II) $x * e = x$
(BH) $x * y = e$ and $y * x = e$ imply that $x = y$
(BZ)$((x * z) * (y * z)) * (x * y) = e$

Definition 2.10. [1] Let X be a non-empty set equipped with a binary operation $*$ and fixed element e. Then the algebraic structure $(X, *, e)$ is said to be a QS – algebra, if it satisfies the following axioms, for all $x, y, z \in X$
(I) $x * x = e$
(II) $x * e = x$
(Q) $(x * y) * z = (x * z) * y$
(BM) $(z * x) * (z * y) = y * x$

Definition 2.11. [12] Let X be a non-empty set equipped with a binary operation $*$ and fixed element e. Then the algebraic structure $(X, *, e)$ is said to be a B – algebra, if it satisfies the following axioms, for all $x, y, z \in X$.
(I) $x * x = e$
(II) $x * e = x$
(B) $(x * y) * z = x * (z * (e * y))$

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Definition 2.12. [11] Let X be a non-empty set equipped with a binary operation $*$ and fixed element e. Then the algebraic structure $(X, *, e)$ is said to be a Q – algebra, if it satisfies the following axioms, for all $x, y, z \in X$

(I) $x * x = e$

(II) $x * e = x$

(Q) $(x * y) * z = (x * z) * y$

Definition 2.13. Let X be a non-empty set equipped with a binary operation $*$ and fixed element e. Then the algebraic structure $(X, *, e)$ is said to be **e – commutative** if it satisfies the axiom $x * (e * y) = y * (e * x)$, for all $x, y \in X$.

Notations. Throughout this article, authors used the following notations, for all x, y, z $\in$ X.

(D): $e * (e * x) = x$

(E): $x * (e * y) = y * (e * x)$

(F): $y * (y * x) = x$

(G): $(e * x) * (e * y) = e * (x * y) = y * x$

Example 2.14. Let X = {0, 1, 2} and $*$ be the binary operation defined on X as shown in the following table.

$*$	0	1	2
0	0	1	2
1	1	0	2
2	2	2	0

Then (X,$*$, 0) is a BF – algebra and BH – Algebra, but not a BG – algebra. Hence, (X,$*$, 0) is a BF_2 – algebra.

Example 2.15. Let X = {0, 1, 2} and $*$ be the binary operation defined on X as shown in the following table.

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

Then (X,$*$, 0) is a BG – algebra and BH – algebra, but not BF – algebra.

Theorem 2.16**.** Every BG – algebra is a BH – algebra, but not conversely.

Proof. Let (X,$*$, e) is a BG – algebra and suppose that x $*$ y = e, for all x, y $\in$ X.

We have, x = (x $*$ y) $*$ (e $*$ y). (by BG)

$\Rightarrow$ x = e $*$ (e $*$ y) (by D)

$\Rightarrow$ x = y

i.e., x $*$ y = e $\Rightarrow$ x = y, for all x, y $\in$ X.

Also y $*$ x = e = x $*$ x (by I)

$\Rightarrow$ y = x (Refer [9])

Hence, x $*$ y = e = y $*$ x = e implies that x = y, for all x, y $\in$ X.

Converse of the above statement is not true in general.
From example 2.14, $(x * y) * (0 * y) \neq x$, for $x = 1$, $y = 2$.
Hence, every BH – algebra need not be a BG – algebra.

Example 2.17. Let $X = \{0, 1, 2\}$ and $*$ be the binary operation defined on X as shown in the following table.

$*$	0	1	2
0	0	1	2
1	1	0	2
2	2	1	0

Then $(X,*, 0)$ is a BH – algebra, but not BF – algebra and BG – algebra.

Theorem 2.18. Every BF_1 – algebra is a BF_2 – algebra, but not conversely.

Proof. Let $(X,*, e)$, for any fixed $e \in X$ be a BF_1 – algebra. Then it is enough to prove that (BH) also holds well.
Suppose that $x * y = e$, for all $x, y \in X$.
$(BG) \Rightarrow x = (x * y) * (e * y) = e * (e * y) = y \Rightarrow x = y$.
i.e., $x * y = e \Rightarrow x = y$, for all $x, y \in X$.
Again, suppose that $y * x = e \Rightarrow x * y = e * (y * x) = e * e = e \Rightarrow x * y = e \Rightarrow x = y$.
i.e., $y * x = e \Rightarrow x = y$, for all $x, y \in X$.
Hence, $x * y = e = y * x$ implies that $x = y$, for all $x, y \in X$.
Therefore, every BF_1 – algebra is a BF_2 – algebra.
Converse of the above statement need not be true.
From example 2.14, it is clear that $(x * y) * (0 * y) \neq x$ for $x =1$, $y = 2$.
i.e., $(X,*, 0)$ is not a BG – algebra.
Hence, $(X,*, 0)$ is a BF_2 – algebra but not a BF_1 – algebra.

Corollary 2.19. Every e – Commutative BF_1 – Algebra is a BF_2 – algebra, but not conversely.
Proof. From Theorem 2.18, Every BF_1 – algebra is a BF_2 – algebra and hence every e – Commutative BF_1 – algebra is a BF_2 – algebra.
Converse of the above statement need not be true.

Example 2.20. Let $X = \{0, 1, 2\}$ and $*$ be the binary operation defined on X as shown in the following table.

$*$	0	1	2
0	0	1	2
1	1	0	0
2	2	0	0

But, $(x * y) * (0 * y) \neq x$ for $x =1$, $y = 2$.
i.e., $(X,*, 0)$ is not a BG – algebra and hence not a BF_1 – algebra and hence not a 0 – Commutative BF_1 – algebra.
Therefore, every BF_2 – algebra need not be 0 – Commutative BF_1 – algebra.

Note: From the above example it is clear that,

(i) Every BF_2 – algebra need not be BG – algebra.

(ii) Every 0 – commutative BF_2 – algebra need not be a BG – algebra.

Example 2.21. Let Z denotes the set of all Integers. Define a binary operation $*$ on Z such that $x * y = x - y$, for all $x, y \in Z$. Then $(Z,*, -)$ is a 0 – commutative BF_1 – algebra and 0 – commutative BF_2 – algebra.

Example.2.22. Let R denotes the set of all Real numbers. Define a binary operation $*$ on R such that $x * y = x - y$, for all $x, y \in R$. Then $(R,*, 0)$ is a 0 – commutative BF_1 – algebra and 0 – commutative BF_2 – algebra.

Example 2.23. Let R denotes the set of all Real numbers. Define a binary operation $*$ on R such that $x * y = x - y + \sqrt{n}$, $n \geq 0$ for all $x, y \in R$. Then $(R,*, \sqrt{n})$ is a $\sqrt{n}$ – commutative BF_1 – algebra and hence $\sqrt{n}$ – commutative BF_2 – algebra.

Theorem 2.24. Every BP – algebra is a BF – algebra, but not conversely.

Proof. Let $(X,*, e)$ is a BP – algebra. Then (I) holds well. Now it is enough to prove that (II) and (BF) also holds well.

(II): $x * e = x * (x * x)$ (by I)
$= x$ (by BP_1)

(BF): $e * (y * x) = (x * x) * (y * x)$ (by I)
$= x * y$ (by BP_2)

Hence, $(X,*, e)$ is a BF – algebra.
Converse of the above statement is not true in general. From example 2.14, $(X,*, 0)$ is a BF – algebra, but not a BP– algebra, as $(x * z) * (y * z) \neq x * y$, for $x = 0$, $y = 1$. $z = 2$.
Therefore, every BF – algebra need not be a BP – algebra.

Theorem 2.25. Every BP – algebra is a BG – algebra, but not conversely.

Proof. Let $(X,*, e)$ is a BP – algebra. Then from theorem 2.24, (II) holds well. Now it is enough to prove that (BG) also holds well.

(BG): $(x * z) * (e * z) = x * e$ (by BP_2)
$= x$ (by II)

Hence, $(X,*, e)$ is a BG – algebra.
Converse of the above statement is not true in general.
From example 2.15, $(X,*, 0)$ is a BG – algebra, but not a BP – algebra, as $(x * z) * (y * z) \neq x * y$, for $x = 0$, $y = 1$, $z = 2$.
Therefore, every BG – algebra need not be BP – algebra.

Theorem 2.26. Every BP – algebra is a BH – algebra, but not conversely.

Proof. Let $(X,*, e)$ is a BP – algebra. Then from theorem 2.24, (II) holds well. Now it is enough to prove that (BH) also holds well.
Suppose that $x * y = e$, for all $x, y \in X$.
(BP_1): $x * (x * y) = y \Rightarrow x * (e) = y \Rightarrow x = y$.
i.e., $x * y = e \Rightarrow x = y$ for all $x, y \in X$.
Again, if $y * x = e$, for all $x, y \in X$, then $x * y = e * (y * x) = e * (e) = e$.
i.e., $x * y = e \Rightarrow x = y$, for all $x, y \in X$.
Hence, $x * y = e = y * x$ implies that $x = y$, for all $x, y \in X$.
Hence every BP – algebra is a BH – algebra.

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Converse of the above statement is not true in general. Refer the following example.

Example 2.27. Let X= {0, 1, 2} and $*$ be the binary operation defined on X as shown in the following table.

$*$	0	1	2
0	0	1	1
1	1	0	2
2	2	2	0

Clearly, $(X,*, 0)$ is a BH – algebra, but not BP – algebra, as $(x * z) * (y * z) \neq x * y$, for x = 1, y = 0 and z = 2. Hence, every BH – algebra need not be a BP – algebra.

Theorem 2.28. Let $(X,*, e)$ BP – algebra. Then X is a BF_1 – algebra.

Proof. Since every BP – algebra is a BF – algebra and BG – algebra, then every BP – algebra is a BF_1 – algebra.

Theorem 2.29. Every BP – algebra is a BF_2 – algebra, but not conversely.

Proof. Since every BP – Algebra is BF – algebra and BH – algebra then every BP – algebra is a BF_2 – algebra. Converse of the above statement need not be true.

Example 2.30. Let X= {0, 1, 2} and $*$ be the binary operation defined on X as shown in the following table.

$*$	0	1	2
0	0	2	1
1	1	0	0
2	2	0	0

Clearly, $(X,*, 0)$ is a BF_2 – algebra, but not BP – Algebra, as $(x * z) * (y * z) \neq x * y$, for x = 1, y = 0 and z = 2 and also $y * (y * x) \neq x$, for x = 2, y = 1.
Therefore, every BP – algebra need not be a BF_2 – algebra.

Theorem 2.31. Let $(X,*, e)$ be a BG – algebra with the condition that $z = x * y$, for all $x, y, z \in X$. Then $(X,*, e)$ is a B – algebra.

Proof. Since $(X,*, e)$ is a BG – algebra then it is enough to prove that (B) also holds good.

Consider, (B): $(x * y) * z = (x * y) * (x * y)$ (since $z = x * y$)
$= e$ (by I)
$= x * x$ (by I)
$= x * ((x * y) * (e * y))$ (by BG)
$= x * (z * (e * y))$, (by $z = x * y$)

Hence, $(x * y) * z = x * (z * (e * y))$, for all $x, y, z \in X$.
Therefore, $(X,*, e)$ is a B – algebra.

Corollary 2.32. Let $(X,*, e)$ be a BF_1 – algebra with the condition that $z = x * y$, for all $x, y, z \in X$. Then X is a B – algebra.

Corollary 2.33. Let (X,∗, e) be an e – commutative BF_1 – algebra with the condition that z = x ∗ y. Then X is a B – algebra.

Theorem 2.34. Let (X,∗, e) be a BG – algebra with the condition that z = e ∗ y, for all y, z ∈ X. Then X is a B – algebra.

Proof. Since (X,∗, e) is a BG – algebra, then it is enough to prove that (B) also holds good.

Consider, (B): (x ∗ y) ∗ z = (x ∗ y) ∗ (e ∗ y) (since z = e ∗ y)
= x (by BG)
= x ∗ e (by II)
= x ∗ ((e ∗ y) ∗ (e ∗ y)) (by I)
= x ∗ (z ∗ (e ∗ y)) (since z = e ∗ y)

Hence, (x ∗ y) ∗ z = x ∗ (z ∗ (e ∗ y)), for all x, y, z ∈ X.
Therefore, (X,∗, e) is a B – algebra.

Corollary *2.35.* Let (X,∗, e) be a BF_1 – algebra with the condition that z = e ∗ y, for all y, z ∈ X. Then (X,∗, e) is a B – algebra.

Corollary 2.36. Let (X,∗, e) be an e – commutative BF_1 – algebra with the condition that z = e ∗ y, for all x, y, z ∈ X. Then (X,∗, e) is a B – algebra.

Theorem 2.37. Let (X,∗, e) be a BG – algebra with z = e ∗ y, for all y, z ∈ X. Then X is a Q – algebra.

Proof. Since X is a BG – algebra, then it is enough to prove that (Q) holds well.

Consider, (Q): (x ∗ y) ∗ z = (x ∗ y) ∗ (e ∗ y) (since z = e ∗ y)
= x (by BG)
= (x ∗ (e ∗ y)) ∗ (e ∗ (e ∗ y)) (by BG)
= (x ∗ (e ∗ y)) ∗ y (by BG)
= (x ∗ z) ∗ y (since z = e ∗ y)

Hence, (x ∗ y) ∗ z = (x ∗ z) ∗ y, for all x, y, z ∈ X.
Therefore, (X,∗, e) is a Q – algebra.

Corollary 2.38. Let (X,∗, e) be an e – commutative BG – algebra with z = e ∗ y, for all y, z ∈ X. Then X is a Q – algebra.

Corollary 2.39. Let (X,∗, e) be a BF_1 – algebra with z = e ∗ y, for all y, z ∈ X. Then X is a Q – algebra.

Corollary 2.40. Let (X,∗, e) be an e **-** commutative BF_1 – algebra with z = e ∗ y, for all y, z ∈ X. Then X is a Q – algebra.

Theorem 2.41. Let (X,∗, e) be an e – commutative BF_1 – algebra with z = x ∗ y, for all x, y, z ∈ X. Then X is a Q – algebra.

Proof. Since X be an e – commutative BF_1 – algebra, then it is enough to prove that (Q) holds well.

Consider, (Q): (x ∗ y) ∗ z = (x ∗ y) ∗ (x ∗ y) (since z = x ∗ y)
= e (by I)
= y ∗ y (by I)
= ((y ∗x) ∗ (e ∗ x)) ∗ y (by BG)
= (x ∗ (e ∗ (y ∗x))) ∗ y (by E)
= (x ∗ (x ∗y)) ∗ y (by BF)
= (x ∗ z) ∗ y (since z = x ∗ y)

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Hence, $(x * y) * z = (x * z) * y$, for all $x, y, z \in X$.
Therefore, $(X,*, e)$ is a Q – algebra.

Theorem 2.42. Let $(X,*, e)$ be a BF_1 – algebra with $z = x * y$, for all $x, y, z \in X$ and (G). Then X is a Q – algebra.

Proof: Since X is a BF_1 – algebra with $z = x * y$, for all $x, y, z \in X$ and (G), then it is enough to prove that (Q) holds well.

Consider, (Q): $(x * y) * z = (x * y) * (x * y)$ (since $z = x * y$)
$= e$ (by I)
$= y * y$ (by I)
$= ((y * x) * (e * x)) * y$ (by BG)
$= (e * ((e * x) * (y * x))) * y$ (by BF)
$= ((e * (e * x)) * (e * (y * x))) * y$ (by G)
$= (x * (x * y)) * y$ (by D and BF)
$= (x * z) * y$ (since $z = x * y$)

Hence, $(x * y) * z = (x * z) * y$, for all $x, y, z \in X$.
Therefore, $(X,*, e)$ is a Q – algebra.

Theorem 2.43. Let $(X,*, e)$ be a BF – algebra with the condition that $z = x * y$, for all $x, y, z \in X$. Then X is a BN – algebra.

Proof. Since X is a BF – algebra then it is enough to prove that (BN) holds good.

Consider, (BN): $(x * y) * z = (x * y) * (x * y)$ (since $z = x * y$)
$= e$ (by I)
$= (y * x) * (y * x)$ (by I)
$= (e * (x * y)) * (y * x)$ (by BF)
$= (e * z) * (y * x)$ (since $z = x * y$)

Hence, $(x * y) * z = (e * z) * (y * x)$, for all $x, y, z \in X$.
Therefore, $(X,*, e)$ is a BN – algebra.

Corollary 2.44. Let $(X,*, e)$ be an e – commutative BF – algebra with $z = x * y$, for all $x, y, z \in X$. Then X is a BN – algebra.

Corollary 2.45. Let $(X,*, e)$ be a BF_1 – algebra with $z = x * y$, for all $x, y, z \in X$. Then X is a BN – algebra.

Corollary 2.46. Let $(X,*, e)$ be a BF_2 – algebra with $z = x * y$, for all $x, y, z \in X$. Then X is a BN – algebra.

Corollary 2.47. Let $(X,*, e)$ be an e – commutative BF_1 – algebra with $z = x * y$, for all $x, y, z \in X$. Then X is a BN – algebra.

Corollary 2.48. Let $(X,*, e)$ be an e – commutative BF_2 – algebra with $z = x * y$, for all $x, y, z \in X$. Then X is a BN – algebra.

Theorem 2.49. Let $(X,*, e)$ be an e – commutative BF_1 – algebra with $z = e * y$, for all $y, z \in X$. Then X is a BN – algebra.

Proof. Taking $z = e*y$ and using the conditions (BG), (F) and (D) easily any one can prove this theorem.

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Theorem 2.50. Let (X,∗, e) be an e – commutative BF – algebra. Then X is a BN – algebra.

Proof. By using (D), (E) and (BF) we can prove this theorem.

Corollary 2.51. Let (X,∗, e) be an e – commutative BF_1 – algebra. Then X is a BN – algebra.

Corollary 2.52. Let (X,∗, e) be an e – commutative BF_2 – algebra. Then X is a BN – algebra.

Theorem 2.53. Let (X,∗, e) be a BF – algebra with (G). Then X is a BN – algebra.

Proof. Proof is straight forward.
Corollary 2.54. Let (X,∗, e) be a BF_1 – algebra with (G). Then X is a BN – algebra.

Corollary 2.55. Let (X,∗, e) be a BF_2 – algebra with (G). Then X is a BN – algebra.

Proposition 2.56. Let (X,∗, e) be a BM – algebra. Then for all x, y ∈ X, the following are true.
(i) $e * e = e$
(ii) $x * x = e$
(D) $e * (e * x) = x$
(E) $x * (e * y) = y * (e * x)$
(F) $x * (x * y) = y$
(G) $(e * x) * (e * y) = y * x = e * (x * y)$

Proof. Proof is straight forward.

Theorem 2.57. Every BM – algebra is a BF – algebra, but not conversely.
Converse of the above statement need not be true.

Example 2.58. Let X = {0, 1, 2} and ∗ be the binary operation defined on X as shown in the following table.

∗	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

Then (X,∗, 0) is a BF – algebra, but not a BM – algebra, as $(z * x) * (z * y) \neq y * x$, for x = 2, y = 0 and z = 1.
Therefore, every BF – algebra need not be a BM – algebra.

Theorem 2.59. Every BM – algebra is a BG – algebra, but not conversely.

Proof. By using the conditions (BF), (G), (BM) and (II), we prove this theorem.
Converse of the above statement need not be true.
From example 2.14, $(z * x) * (z * y) \neq y * x$, for x = 0, y = 1, z = 2.
Therefore, every BG – algebra need not be a BM – algebra.

Theorem 2.60. Let (X,∗, e) is a BM – algebra. Then X is a BF_1 – algebra.

Theorem 2.61. Every BM – algebra is a BH – algebra, but not conversely.

Proof. Proof is straight forward.
Converse of the above statement need not be true.

Example 2.62. Let X= {0, 1, 2} and $*$ be the binary operation defined on X as shown in the following table.

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	2	0

Then $(X,*, 0)$ is a BH – algebra but not a BM – algebra, as $(z * x) * (z * y) \neq y * x$, for $x = 0$, $y = 1$ and $z = 2$.
Therefore, every BH – algebra need not be a BM – algebra.

Theorem 2.63. Every BM – algebra is a BF_2 – algebra but not conversely.

Proof: Since every BM – algebra is a BF – algebra and BH – algebra then every BM – algebra is a BF_2 – algebra.
Converse of the above statement need not be true. Refer the following example.

Example 2.64. Let X= {0, 1, 2} and $*$ be the binary operation defined on X as shown in the following table.

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	1
3	3	2	3	0

Clearly, $(X,*, 0)$ is a BF_2 – algebra but not a BM – algebra, as $(z * x) * (z * y) \neq y * x$, for $x = 2$, $y = 0$ and $z = 3$.
Therefore, every BF_2 – algebra need not be a BM – algebra.

Theorem 2.65. Cancellation laws holds good in BM – algebra.

Proof. Let $(X,*, e)$ be a BM – algebra.
Left Cancellation Law: $z * x = z * y \Rightarrow x = y$, for all $x, y, z \in X$.
Suppose that, $z * x = z * y$
From (BM), $(z * x) * (z * y) = y * x$
$\Rightarrow (z * y) * (z * y) = y * x$
$\Rightarrow e = y * x$ (by II)
Since, $x * y = e * (y * x) = e * e = e \Rightarrow x * y = e$
Since every BM – algebra is a BG – algebra, then $x = (x * y) * (e * y) = e * (e * y) = y \Rightarrow x = y$.
Thus, left Cancellation Law holds good in BM – algebra.
Right Cancellation Law: $x * z = y * z \Rightarrow x = y$, for all $x, y, z \in X$.
Suppose that $x * z = y * z$

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$\Rightarrow e * (x * z) = e * (y * z)$

$\Rightarrow z * x = z * y$ (by BF)

$\Rightarrow x = y$ (by LCL)

Therefore, right Cancellation Law holds good in BM – algebra.

Theorem 2.66. Let (X,∗, e) be an e - commutative BF_1 – algebra with $z = x * y$, for all x, y, z ∈ X. Then (X,∗, e) is a BM – algebra.

Proof. Since (X,∗, e) is an e - commutative BF_1 – algebra then (II) holds good. Now it is enough to prove that (BM): $(z * x) * (z * y) = y * x$, for all x, y, z ∈ X, also holds well.

Consider, $(z * x) * (z * y) = (z * x) * (e * (y * z))$

$= (y * z) * (e * (z * x))$ (by E)

$= (y * z) * (e * (e * y))$ (since $z * x = e * y$)

$= (y * z) * y$ (by D)

$= e * (y * (y * z))$ (by BF)

$= e * z$ (by F)

$= y * x$ (by BF)

Hence, $(z * x) * (z * y) = y * x$, for all x, y, z ∈ X.

Therefore, (X,∗, e) is a BM – algebra.

Corollary 2.67. Let (X,∗, e) be an e – commutative BF_1 – algebra with $z = x * y$, for all x, y, z ∈ X. Then (X,∗, e) is a QS – algebra.

Theorem 2.68. Let (X,∗, e) be a BF_1 – algebra with $z = x * y$, for all x, y, z ∈ X and (G). Then (X,∗, e) is a BM – algebra.

Proof. Taking $z = x*y$ and using (BF), (G) and (F) we can prove this theorem.

Corollary .2.69. Let (X,∗, e) be a BF_1 – algebra with $z = x * y$, for all x, y, z ∈ X and (G). Then (X,∗, e) is a QS – algebra.

Theorem 2.70. Let (X,∗, e) be an e - commutative BF_1 – algebra with $z = x * y$, for all x, y, z ∈ X. Then (X,∗, e) is a BM – algebra.

Proof. Proof is similar to proof of theorem 2.69.

Theorem 2.71. Let (X,∗, e) be an e – commutative BF_1 – algebra with $e * z = x * z$, for all x, z ∈ X. Then X is a BM – algebra.

Proof. Since (X,∗, e) is an e – commutative BF_1 – Algebra, then (II) holds good. It is enough to prove that (BM): $(z * x) * (z * y) = y * x$, for all x, y, z ∈ X also holds good.

Consider, $(z * x) * (z * y) = (e * (x * z)) * (z * y)$ (by BF)

$= (e * (e * z)) * (z * y)$ (since $x * z = e * z$)

$= z * (z * y)$ (by D)

$= y * e$ (by II)

$= y * (z * z)$ (by I)

$= y * (z * (z * x))$ (since $x * z = e * z$)

$= y * x$ (by G)

Hence, $(z * x) * (z * y) = y * x$, for all x, y, z ∈ X

Therefore, (X,∗, e) is a BM – algebra.

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Corollary 2.72. Let $(X,*, e)$ is a BF_1 – algebra with (F) and $e * z = x * z$, for all $x, z \in X$. Then X is a BM – algebra.

Theorem 2.73. Let $(X,*, e)$ be an e – commutative BF_1 – algebra with $z * y = e * x$, for all $x, y, z \in X$. Then X is a BM – algebra.

Proof. Proof is similar to proof of the theorem 2.71.

Theorem 2.74. Let $(X,*, e)$ is a BF_1 – algebra with (G) and $z * y = e * x$, for all $x, y, z \in X$. Then X is a BM – algebra.

Proof. By taking $z * y = e * x$ and using (BF), (G), (D) and (F), we can prove theorem 2.74.

Theorem 2.75. Let $(X,*, e)$ is an e – commutative BF – algebra with $(x * z) * (y * z) = x * y$, for all $x, y, z \in X$. Then $(X,*, e)$ is a BM – algebra.

Proof. Taking $(x * z) * (y * z) = x * y$ and using (BF) and (E) we prove this theorem.

Theorem 2.76. Let $(X,*, e)$ be a BP – algebra with $x * z = e$, for all $x, z \in X$. Then X is a BZ – algebra.

Proof. Let $x * z = e$.
Consider, for any $x, y, z \in X$

$$
\begin{aligned}
((x * z) * (y * z)) * (x * y) &= ((e) * (y * z)) * (x * y) && \text{(since } x * z = e\text{)}\\
&= (z * y) * (x * y) && \text{(by BF)}\\
&= z * x && \text{(by } BP_2\text{)}\\
&= e * (x * z) && \text{(by BF)}\\
&= e * e && \text{(since } x * z = e\text{)}\\
&= e && \text{(by II)}
\end{aligned}
$$

Hence, $((x * z) * (y * z)) * (x * y) = e$, for all $x, y, z \in X$.
Therefore, $(X,*, e)$ is a BZ – algebra.

Theorem 2.77. Let $(X,*, e)$ be a BP – algebra with $x * y = e$, for all $x, y \in X$. Then X is a BZ – algebra.

Proof. Let $(X,*, e)$ be a BP – algebra with $x * y = e$, for all $x, y \in X$.
Consider for all $x, y, z \in X$,

$$
\begin{aligned}
((x * z) * (y * z)) * (x * y) &= ((x * z) * (y * z)) * (e) && \text{(since } x * y = e\text{)}\\
&= (x * z) * (y * z) && \text{(by II)}\\
&= x * y && \text{(by } BP_2\text{)}\\
&= e && \text{(since } x * y = e\text{)}
\end{aligned}
$$

Hence, $(X,*, e)$ is a BZ – algebra.

Theorem 2.78. Let $(X,*, e)$ be a BP – algebra with (E). Then X is a BM – algebra.

Proof. Using (I), (BP_1), (BF), (E) and (BP_2) any one can prove this theorem.

Theorem 2.79. Let $(X,*, e)$ be a BP – algebra with (G). Then X is a BM – algebra.

Proof. It is enough to prove that (BM) holds good.
Consider,

$$
\begin{aligned}
(z * x) * (z * y) &= e * ((z * y) * (z * x)) && \text{(by BF)}\\
&= (e * (z * y)) * (e * (z * x)) && \text{(by G)}
\end{aligned}
$$

$= (y * z) * (x * z)$ (by BF)
$= y * x$ (by BP_2)

Hence, every BP – algebra with (G) is a BM – algebra.

Theorem 2.80. Let $(X,*, e)$ is a BM – algebra. Then X is a BP – algebra.

Proof. Proof is straight forward.

Theorem 2.81. Let $(X,*, e)$ be an e – commutative BM – algebra. Then X is a BP – Algebra.

Proof. Let $(X,*, e)$ is an e – commutative BM – Algebra. It is enough to prove that (BP_1): $x * (x * y) = y$ and (BP_2): $(y * z) * (x * z) = y * x$, for all $x, y, z \in X$.

Consider $x * (x * y) = x * (e * (y * x))$ (by BF)
$= (y * x) * (e * x)$ (by E)
$= e * ((e * x) * (y * x))$ (by BF)
$= e * (e * y)$ (by BM)
$= y$ (by D)

Hence, $x * (x * y) = y$, for all $x, y \in X$.

Again consider, (BP_2): $(y * z) * (x * z) = (y * z) * (e * (z * x))$ (by BF)
$= (z * x)) * (e * (y * z))$ (by E)
$= (z * x) * (z * y)$ (by BF)
$= y * x$ (by BM)

Hence, $(y * z) * (x * z) = y * x$, for all $x, y, z \in X$.
Therefore, every e – Commutative BM – algebra is a BP – algebra.

Theorem 2.82. Every BM – algebra with $y * z = e$, for all $y, z \in X$ is a BZ – algebra, but not conversely.

Proof. Let $(X, *, e)$ is a BM – Algebra.
Now, it is enough to prove that $((x * z) * (y * z)) * (x * y) = e$, for all $x, y, z \in X$.

Consider, $((x * z) * (y * z)) * (x * y) = ((x * z) * (e)) * (x * y)$ (since $x * z = e$)
$= (x * z) * (x * y)$ (by II)
$= y * z$ (by BM)
$= e$ (since $y * z = e$)

Therefore, $(X,*, e)$ is a BZ – algebra.

Corollary 2.83. Every BM – algebra with $x * z = e$, for all $x, z \in X$ is a BZ – algebra, but not conversely.

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